MEMORANDUM
RM-4735-PR
SEPTEMBER 1965

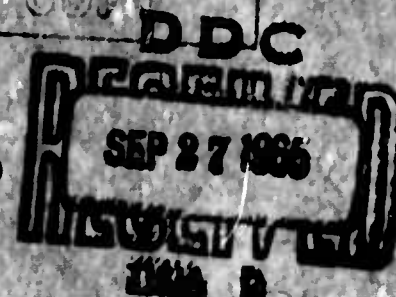# LINEAR AND NONLINEAR
FILTERING THEORY

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PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA · CALIFORNIA

LINEAR AND NONLINEAR
FILTERING THEORY

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PREFACE

Linear and nonlinear filtering has had and will have a profound effect on many problems of the Air Force. In particular, linear filtering theory has been applied to orbit determination, aircraft and missile guidance, and control problems.

ABSTRACT

This paper is a written version of a talk to be given on problems of filtering theory. It is a review of the linear theory and some of the recent fundamental advances in the nonlinear theory.

Some open and quite interesting problems in the non-linear theory are discussed.

CONTENTS

LINEAR AND NONLINEAR FILTERING THEORY

1. INTRODUCTION

The general filtering problem can be described as follows: a stochastic process $z(t)$ is observed consisting of a stochastic process $x(t)$ (i.e. the signal process) corrupted by noise $v(t)$, which is another stochastic process. It is desired to construct a stochastic process $\hat{x}(t)$, an estimate of the signal. Usually only the second-order properties of the signal and noise processes are given.

In the 1940's, Wiener in [1] and Kolmogowv solved this problem under the assumption that the signal and noise processes are second-order stationary processes and that the observations are known for the infinite past, by finding the minimum variance unbiased linear estimate. Here two problems are discussed that generalize the Wiener problem. The first is linear filtering of vector nonstationary second-order processes with partial observations. A fairly complete theory exists (see [2]) and is about five years old. The second problem, that of filtering with the signal (the solution of an arbitrary random differential equation) has a far-from-complete theory as yet, although interesting contributions to this problem have been made lately. The purpose of discussing these two problems is twofold: primarily to point out a research area with many interesting and difficult problems, and

secondly to point out some results that do not seem to have the attention they deserve, except in the area of control theory.

2. THE LINEAR PROBLEM

We shall assume that the signal process x is the solution to the following random differential equation:

$$dx = Fx\ dt + G\ d\beta,$$

$$x(t_0) = c,$$

where x is an n–vector, β is r–dimensional brownian motion, and c is a gaussian zero mean random vector independent of β. Further, F is an n × n matrix, and G is an n × r matrix. The observations will be given as the solution to

$$dz = Hx\ dt + dv,$$

with v an s–dimensional brownian motion independent of β and c, and H an s × n matrix. Further, we assume

$$E[\beta(t) - \beta(s)][\beta(t) - \beta(s)]' = Q(t - s)$$

and

$$E[v(t) - v(s)][v(t) - v(s)]' = R(t - s).$$

Our problem now consists in finding the minimum variance unbiased estimator $\hat{x}(t)$ of x(t) that is a functional of the observations z(s), $t_0 \le s \le t$. It is very easy to describe the desired estimate in this case. Let F_t be the minimal σ–field induced by z(s) for s, $t_0 \le s \le t$. Then

$$\hat{x}(t) = E[x(t)|F_t]$$

so that our problem can be resolved by finding the conditional distribution of $x(t)$, given F_t. But with our assumptions of gaussianess, a little thought reveals that this is $N(L_t(z.), P_t)$, where $L_t(z.)$ is the projection of $x(t)$ on the closed linear subspace of the Hilbert space of L^2 random variables. $P_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$; that is, the conditional covariance is identical with the unconditional covariance—a direct consequence of gaussianess. $\hat{x}$ is a projection form of the Wiener–Hopf equation, namely,

$$(2.1) \qquad [x(t) - \hat{x}(t), u] = E\widetilde{x}(t)u' = 0$$

for every n–vector u in the subspace determined by the observations. In fact, by letting $\hat{x}(t) = \int_{t_0}^{t} W(t, \tau) \, dz_\tau$, Eq. (2.1) becomes

$$(2.2) \quad \text{cov } x(t), \, Hx(s) = \int_{t_0}^{t} W(t, \tau) \, \text{cov } [Hx(\tau), \, Hx(s)] \, d\tau + RW(t, t),$$

the matrix form of the Wiener–Hopf equation. Rather than solve Eq. (2.2) directly, it is more convenient to give $\hat{x}(t)$ by a sequential recipe. Now Eq. (2.2) can be used to derive the following dynamical evolution equations for $\hat{x}$ and P [see Eq. (2.2)]:

$$d\hat{x} = F\hat{x} \, dt + PH'R^{-1}(dz - H\hat{x} \, dt),$$
(2.3)
$$\hat{x}(t_0) = 0$$

$$\frac{dP}{dt} = FP + PF' - PH'R^{-1}HP + GQG',$$

(2.4)

$$P(t_0) = ECC'.$$

At this point we note that Eqs. (2.3) and (2.4) have solutions that determine minimum variance <u>linear</u> unbiased istimator if the gaussian assumption is dropped, since, in the gaussian case, the projection and the condition mean coincide.

To ensure that Eqs. (2.3) and (2.4) are computationally effective schemes for producing our estimate, it becomes necessary to ensure that they are stable and investigate their asymptotic behavior. Toward this end we define observability and controllability as follows.

<u>Definition</u>. The model is completely controllable (observable) if for every $t_0(t)$ there exists a $t > t_0$ $(t_0 < t)$ such that

$$\int_{t_0}^{t} e^{F(t-s)} GG' e^{F'(t-s)} ds = C(t_0, t)$$

$$\int_{t_0}^{t} e^{F'(s-t)} H'He^{F(s-t)} ds = M(t_0, t)$$

is positive definite. A differential equation is said to be stable if solutions initiating at any time close to the equilibrium solution stay close to that solution for all time. It is said to be asymptotically stable if, further, these solutions approach the equilibrium solution as time tends to infinity. It is said to be uniformly asymptotically

stable if it is asymptotically stable uniformly in time and in the initial conditions.

Theorem. If the model is completely observable and if $\Pi(t, t_0, 0)$ is the solution of Eq. (2.4), with $P(t_0) = 0$, then

$$\bar{P} = \lim_{t_0 \to -\infty} \Pi(t, t_0, 0)$$

exists and is a solution of Eq. (2.4). Further, if the model is also completely controllable, then Eq. (2.3) is uniformily asymptotically stable and

$$\bar{P} = \lim_{t_0 \to -\infty} \Pi(t, t_0, P_0)$$

for any P_0 semidefinite, and Eq. (2.4) is asymptotically stable. Still further, Eq. (2.3) with $P = \bar{P}$ is the Wiener optimal filter.

As the previous theorem makes abundantly clear, controllability and observability are fundamental, but they also have interesting intuitive meanings, namely, if $\dot{x} = Fx$ and $z = Hx$, and one knows F and $z(\cdot)$ on $[t_0, t]$, then $x(t_0)$ is determined when the system is observable. Further, $W(t_0, t)$ is just the Fisher information matrix for estimating initial conditions of a linear differential equation. Controllability suffices so that every two points of R^n may be connected by motions of $\dot{x} = Fx + Gu$ with appropriate u's. As one might suspect by the names chosen for our conditions, there is a complete duality between linear filtering and control problems. In fact,

the methods of proof of the above theorem rely on this duality. Finally, everything previously said generalizes to the nonstationary case, i.e., F, G, H, R, Q functions of time.

3. NONLINEAR FILTERING

As we have seen in the linear gaussian case, the central problem in filtering theory is the determination of the conditional distribution of the signal $x(t)$, given the σ-field F_t determined by the observations $z(s)$ for $s \in (t_0, t)$. We shall now generalize the model of the signal and observation mechanisms as

$$(3.1) \qquad dx = f(x, t)\, dt + \sigma(x, t)\, d\beta,$$

$$(3.2) \qquad dz = h(x, t)\, dt + dv.$$

For this problem, which we shall call the nonlinear filtering problem, Stratonovich [3] was the first to devise a method to obtain the requisite conditional distribution. Unfortunately his final equation was not correct, since he neglected a term. Kushner [4], using the same formal methods as Stratonovich, obtained the correct equation for the conditional distribution. Wonham [5] employed somewhat different methods to solve a special case of this problem. Finally, the author gave, in [6], an abstract representation of the conditional distribution as an integral over function space that leads quickly and easily via random calculus to the correct equations. In order to state our results we introduce the following notation for $g(x, t)$. Here g_t is defined as

$$\hat{g}_t = Eg(x_t, t)|F_t)$$

and is shown to satisfy

$$(3.3) \qquad d\hat{g}_t = (\widehat{Ag})_t \, dt + \left[(\widehat{gh})_t - \hat{h}_t\hat{g}_t\right]R^{-1}(dz_t - \hat{h}_t \, dt),$$

with A the infinitesimal generator of the space-time
Markov process determined by Eq. (3.1). Investigation of
asymptotic behavior of Eq. (3.3) as $t_0 \to -\infty$ and questions
of existence and uniqueness of solutions of Eq. (3.3) are
untouched and interesting. Further conditions for which
the system

$$\dot{x} = f(x),$$

$$x(0) = c,$$

$$z = g(x)$$

is observable—i.e., when the values $z(s)$, $0 < s < \epsilon$, for
every $\epsilon > 0$ determine $x(0)$ uniquely—seem to be quite
difficult, although Markus has introduced local concepts
in a paper on control theory.

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DOCUMENT CONTROL DATA

I. ORIGINATING ACTIVITY	2a. REPORT SECURITY CLASSIFICATION
THE RAND CORPORATION	UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE

LINEAR AND NONLINEAR FILTERING THEORY

4. AUTHOR(S) (Last name, first name, initial)

Bucy, Richard S.

5. REPORT DATE	6a. TOTAL NO. OF PAGES	6b. NO. OF REFS.
September 1965	17	8

7. CONTRACT or GRANT NO.	8. ORIGINATOR'S REPORT NO.
AF 49(638)-700	RM-4735-PR

9a. AVAILABILITY/LIMITATION NOTICES	9b. SPONSORING AGENCY
	United States Air Force Project RAND

10. ABSTRACT	11. KEY WORDS
The text of a talk on problems of filtering theory. It is a review of the linear theory and some of the recent fundamental advances in the nonlinear theory. Open and interesting problems in the nonlinear theory are discussed.	Filtering theory Mathematics